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DYNAMIC INDENTATION OF AN ELASTIC HALF-PLANE BY A RIGID WEDGE: FRICTIONAL AND TANGENTIAL-DISPLACEMENT EFFECTS

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Abstract — A dynamical contact problem is studied in this paper. It involves an elastic half-plane which is indented by a rigid wedge-shaped body. In an effort to depart from the classical formulation of this problem, we consider frictional and tangential-displacement effects. More specifically, it is assumed that Coulomb friction develops between the contacting bodies and also that the tangential surface displacements are not negligible and should thus be coupled with the normal surface displacements in imposing the contact-zone boundary conditions. Certainly, the foregoing considerations model the dynamic indentation of an elastic half-plane in a more realistic way than the usual frictionless and uncoupled formulation. The contact region is assumed to extend at a constant sub-Rayleigh speed (this situation can be achieved by conveniently specifying the indentor kinetics), whereas, due to symmetry, friction may act in opposing directions on opposite sides of the indentor. The study exploits the problem's self-similarity by utilizing homogeneous-function techniques along with the Riemann–Hilbert problem analysis. As the present exact analysis shows, both the sign reversal of the tangential traction and the coupling of the displacement components along the contact length strongly influence the contact-stress behavior at the wedge-apex station. In particular, friction tends to create a power-type singularity at the changeover point of boundary conditions (due to symmetry, this point here is the point where the wedge apex makes contact with the half-plane surface), whereas the tangential-displacement effect tends to eliminate singular behavior there. Representative numerical results are given for the normal stress and tangential displacement along the contact zone, and the relation between the contact-zone expansion velocity and the indentor velocity.

INTRODUCTION

Transient dynamical indentations of elastic half-spaces have been investigated during the last two decades mainly through the efforts of Willis (1973, 1989), Bedding and Willis (1973, 1976), Afanas'ev and Cherepanov (1973), Brock (1976, 1977, 1978, 1979, 1981), Aboudi (1977), Cherepanov (1979), Kawatate (1975), Shukla and Rossmanith (1986), Rossmanith (1987), and Downey and Bogy (1987). Even more recent developments include work done by Georgiadis and Barber (1993), Brock (1993), and Brock and Georgiadis (1994), among others. These rapid-indentation problems enjoy applications in rock, penetration and particulate-media mechanics [see e.g. Altiero and Sikarskie (1975); Cherepanov (1979); Shukla and Rossmanith (1986)].

Typical of such problems is the indentation of an elastic half-plane by a rigid punch, which is rapidly driven into the deformable body so that *stress waves* are generated. The contact region may expand at a velocity on the order of the elastic-wave velocities (therefore, this velocity determines the dynamic character of the problem), whereas the punch velocity may generally be much lower. Important analytical techniques have been developed to deal with these problems, e.g. by Willis (1973), Afanas'ev and Cherepanov (1973), Brock (1976, 1977, 1978), Cherepanov (1979), and Brock (1993). Also, general questions of existence/uniqueness of solutions were dealt with by Georgiadis and Barber (1993).

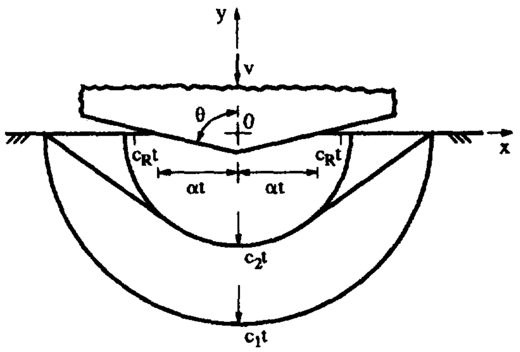

Fig. 1. Dynamic frictional indentation of an elastic half-plane by a uniformly moving rigid wedge. The tangential displacements are not negligible in imposing the contact-zone boundary condition.

In the present work, the well-investigated problem of wedge indentation of a linearly elastic half-plane is re-examined but with the additional consideration of *frictional* and *tangential-displacement* effects. More specifically, as Fig. 1 shows, we consider a symmetrical rigid wedge that begins at time $t = 0$ to indent the half-plane surface through vertical motion at *constant* velocity. The present formulation assumes that: (a) Coulomb friction is generated between the rigid and deformable body and, therefore, the friction- generated shear should act in opposing directions with respect to the origin O, along the expanding contact length, and (b) the tangential surface displacements are not negligible and should thus be coupled with the normal surface displacements in imposing the contact-zone boundary conditions. Thus, our analysis and results depart from the ones based on the classical formulation of the problem [see e.g. Afanas'ev and Cherepanov (1973); Eringen and Suhubi (1975); Bedding and Willis (1976)], which assumes *frictionless* contact and *non-coupling* of the displacement components.

From the viewpoint of mathematical analysis, the present problem is an initial/boundary value problem (IBVP) involving wave equations for the dilatation and rotation under mixed conditions prescribed along *changing* lengths. Problems of this type do not generally lend themselves to a simple analytical treatment. However, the solution is here effected by exploiting the problem's *self-similarity* and utilizing *homogeneous-function* techniques (Brock, 1976, 1977, 1978) and the *Plemelj* and *Riemann–Hilbert* problem analyses (Gakhov, 1966; Roos, 1969). Numerical results for certain field quantities are easily extracted then from the analytical form of solution by simple numerical integration.

Finally, it is noted that in indentation problems, tangential-displacement effects were first considered by Brock (1979, 1981) in dynamical studies, whereas sign reversals of the friction-generated shear traction were considered by Roberts (1970), Brock *et al.* (1993, 1994), Brock (1993), and Brock and Georgiadis (1994) in statical as well as dynamical studies. These previous analyses indicated that: (a) the displacement coupling establishes *non-singular* contact-stress behavior at half-plane points making contact with geometrical discontinuities of the indentor (in contrast with the familiar contact-stress singularities resulting from the standard uncoupled problem formulation) and therefore yields a more natural solution behavior, and (b) the point on the contact zone where the shear traction reverses sign acts as a point of *flux singularity* for the contact stress. Therefore in our problem where *both* effects are included, it is expected (and, indeed, verified by the analysis) that these effects will compete with each other in establishing the stress distribution along the contact zone.

In this connection, the present problem can be considered as a generalization of the dynamical wedge indentation problems studied by Brock (1979), and Brock and Georgiadis (1994), since it assumes the *combined* effects of displacement coupling and friction reversal.

PROBLEM STATEMENT

Consider an isotropic, linearly elastic half-plane $y \leqslant 0$ under plane-strain conditions. As depicted in Fig. 1, assume that this body, which was initially at rest, is disturbed by dynamical and frictional wedge indentation; with contact first occurring at the point $x = 0$, $y = 0$ corresponding to the wedge apex at time $t = 0$. The constant indentor speed V along with dimensional considerations lead us to anticipate a *self-similar* solution in which the *particle velocity* in the body is a *homogeneous* function and the contact region expands at *constant* velocity α [see e.g. Eringen and Suhubi (1975)].

In order for a dynamical state be created in the medium, the velocity α should be on the same order of the characteristic wave velocities. However, the present analysis will be restricted to the regime $\alpha < c_R$, where c_R is the *Rayleigh-wave* velocity (Knowles, 1966; Achenbach, 1973), so as to avoid non-uniqueness of the solution or violation of the *Signorini* contact conditions (Georgiadis and Barber, 1993). Also, we must have $V \ll \alpha$, since small-strain considerations demand that the inclination of the wedge face must obey the condition $(\pi-2\Theta)/2 \ll 1$, where Θ is the half-wedge angle.

The pattern of stress waves generated by the imposed surface displacement is also shown in Fig. 1. The *dilatation* and *rotational* waves are cylindrical and radiate away from the apex at speeds c_1 and c_2, respectively. Because the surface is stress-free outside the contact region, the faster-travelling dilatational wave generates secondary cylindrical rotational (head) waves whose wavefront envelopes define the wedge-like regions shown. In addition, Rayleigh waves propagate along the surface away from the apex.

By utilizing a system of polar coordinates (r, θ) in addition to the (x, y)-system shown in Fig. 1, the governing equations for such a state are written as

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \tag{1a}$$

$$\omega = \frac{1}{2r}\frac{\partial (r \cdot u_\theta)}{\partial r} - \frac{1}{2r}\frac{\partial u_r}{\partial \theta}, \tag{1b}$$

$$\sigma_r = \frac{\mu}{m^2}\left[\frac{\partial u_r}{\partial r} + \frac{1-2m^2}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\right], \tag{2a}$$

$$\sigma_\theta = \frac{\mu}{m^2}\left[\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) + (1-2m^2)\frac{\partial u_r}{\partial r}\right], \tag{2b}$$

$$\sigma_{r\theta} = \mu\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\right], \tag{2c}$$

$$\nabla^2\Delta = \frac{1}{c_1^2}\frac{\partial^2 \Delta}{\partial t^2}, \quad \nabla^2\omega = \frac{1}{c_2^2}\frac{\partial^2 \omega}{\partial t^2}, \tag{3a, b}$$

where Δ is the dilatation, ω is the rotation, (u_r, u_θ) and $(\sigma_r, \sigma_\theta, \sigma_{r\theta})$ are the components of the displacement vector and stress tensor, $c_1 \equiv [(\lambda+2\mu)/\rho]^{1/2}$, $c_2 \equiv (\mu/\rho)^{1/2}$, $m \equiv c_2/c_1$, λ and μ are the Lamé constants of the material, ρ is the mass density, and ∇^2 is the Laplacian operator in the (r, θ)-plane. All field quantities above are functions of the spatial variables r, θ and the time variable t.

Now, the *boundary*, *continuity* and *initial* conditions for the problem are written as

$$u_\theta = -Vt + (r+u_r)\cdot\cot\Theta \quad \text{for } (\theta = 0, r < \alpha t), \text{ and } (\theta = -\pi, r < \alpha t), \tag{4a}$$

$$\sigma_{r\theta} + \operatorname{sgn}(x)\cdot\gamma\cdot\sigma_\theta = 0 \quad \text{for } (\theta = 0, r < \alpha t), \text{ and } (\theta = -\pi, r < \alpha t), \tag{4b}$$

$$\sigma_\theta, \sigma_{r\theta} = 0 \quad \text{for } (\theta = 0, r > \alpha t), \text{ and } (\theta = -\pi, r > \alpha t), \tag{4c}$$

$$\Delta, u_r, u_\theta, \frac{\partial u_r}{\partial t}, \frac{\partial u_\theta}{\partial t} = 0 \quad \text{for } r = c_1 t, -\pi \leqslant \theta \leqslant 0, \tag{5a}$$

$$\omega = 0 \quad \text{for } r = c_2 t/h_2, -\pi \leqslant \theta \leqslant 0, \tag{5b}$$

$$t \leqslant 0 : u_r, u_\theta, \frac{\partial u_r}{\partial t}, \frac{\partial u_\theta}{\partial t} = 0, \tag{6}$$

where sgn( ) is the signum function, γ is the friction coefficient (positive constant), and we have also defined $m_k \equiv c_k/c_1$, $(k = 1,2)$, $h_k \equiv 1$ for $[\cos^{-1}(m_k)-\pi] \leqslant \theta \leqslant -\cos^{-1}(m_k)$ but otherwise $h_k \equiv m_k|\cos\theta| + (1-m_k^2)^{1/2}\sin\theta$.

In the above set of conditions, eqn (4a) expresses the imposed-displacement equality in view also of displacement coupling. For notational simplicity and convenience in treating the uncoupled problem, viz. with u_r being absent in eqn (4a), the latter equation is rewritten as

$$u_\theta + \beta \cdot u_r = -Vt + r \cdot \cot\Theta \quad \text{for } (\theta = 0, r < \alpha t), \quad \text{and} \quad (\theta = -\pi, r < \alpha t), \tag{7}$$

where $\beta \equiv -\cot\,\Theta$ can now be considered a general *coupling constant*. Equation (4b) expresses Coulomb's frictional law, and also accounts for the possible *reversal* of friction-generated shear reflecting thus the problem symmetry, i.e. we assume that the contact shear reverses sign at the wedge apex when σ_θ is itself symmetric. Finally, eqns (5a) and (5b) come from physical considerations, regarding continuity of the fields along wavefronts [see e.g. Brock (1976)].

The solution to the above IBVP must also obey certain *physical constraints*. Specifically, an acceptable solution should satisfy the Signorini contact conditions [see e.g. Fichera (1964); Panagiotopoulos (1985); Barber (1992)], which state that: (a) the normal contact stress is non-tensile, and (b) there is no interpenetration between indentor and half-plane material outside the contact length. For this type of problem, Georgiadis and Barber (1993) showed generally that, as long as the contact-zone velocity is confined to the *sub-Rayleigh* $(\alpha < c_R)$ or *superseismic* $(\alpha > c_1)$ range, the Signorini inequalities are always satisfied by a unique solution. In any event, for particular IBVPs, these inequalities can be checked after obtaining the solution.

Another physical constraint on the solution comes from the particular *friction law* utilized here. Indeed, the Coulomb model implies consistent directions of shear stress and relative slip velocity on the material along the contact zone, i.e.

$$\frac{\partial u_r}{\partial t} > 0 \quad \text{for } (\theta = 0, r < \alpha t), \quad \text{and} \quad (\theta = -\pi, r < \alpha t). \tag{8}$$

We shall next provide an analytical scheme for solving the problem defined by eqns (1)–(7). It will be seen also that this solution satisfies all the aforementioned physical constraints.

HOMOGENEOUS FUNCTION TECHNIQUE

As discussed in the Introduction, the homogeneous-function (or self-similar) technique developed by Brock (1976, 1977, 1978) is utilized. This method makes use of the Busemann–Chaplygin procedure [see e.g. Achenbach (1973); Miles (1960)] and analytic-function theory. Only basic results are presented here.

Consider a general uniformly expanding contact region over the surface of a half-plane, where the imposed normal displacement is a given *polynomial* homogeneous of degree $n \geqslant 1$ in r and t. It can be further shown that the following displacement/stress set in polar coordinates

$$u \equiv u_r + \mathrm{i}u_\theta\,, \quad s_r \equiv \sigma_r - \sigma_\theta + \mathrm{i}2\sigma_{r\theta}\,, \quad s_\theta \equiv \sigma_\theta - \mathrm{i}\sigma_{r\theta}, \tag{9}$$

which formally satisfies quiescent initial conditions, wavefront continuity conditions and the governing elasticity equations in $y \leqslant 0$ ($-\pi \leqslant \theta \leqslant 0$), for a given n, is obtained as

$$(2c_1 n)u = J_1[(vt - r/c_1)U(W_1)] - \mathrm{i}J_2[(vt - r/c_1)U(W_2)], \tag{10}$$

$$(c_1^2/\mu)s_r = -(2c_1^2/\mu)s_\theta + (1-m^2)J_1[2U(W_1) - V(W_1)], \tag{11a}$$

$$(2c_1^2/\mu)s_\theta = J_1[V(W_1)] - \mathrm{i}J_2[V(W_2)], \tag{11b}$$

where

$$U(W_k) = \mathrm{Re}\,[p_k(v)W_k] - \mathrm{i}\,\mathrm{Im}\,(W_k)\,, \quad W_k = \Omega_k(\zeta_k) + \overline{\omega_k(\bar{\zeta}_k)}, \tag{12a}$$

$$V(W_k) = K(v)\,\mathrm{Re}\,[p_k(v)W_k] - \mathrm{i}2\,\mathrm{Im}\,(W_k)\,, \quad K(v) = 2 - v^2/m^2, \tag{12b}$$

and the operator J_k is defined as the real integral

$$[(n-1)!]J_k(f) = \int_{r/c_1 t}^{m_k/h_k} v^{-n-2}(vt - r/c_1)^{n-1} f\, \mathrm{d}v, \tag{13}$$

with $k = 1, 2$ and $m_1 = 1$, $m_2 = m = (c_2/c_1)$. Moreover, the complex variables ζ_k are given by

$$\zeta_k = m_k\,\mathrm{sech}\,[\mathrm{sech}^{-1}(v/m_k) + \mathrm{i}\theta]\,, \quad \zeta_k = m_k \sec[\sec^{-1}(v/m_k) \pm |\theta|], \tag{14a, b}$$

for $v \leqslant m_k$ and $m_k \leqslant v \leqslant (m_k/h_k)$, respectively, where $\pm$ denotes $0 \leqslant |\theta| \leqslant \pi/2$ $(+)$, $\pi/2 \leqslant |\theta| \leqslant \pi$ $(-)$. The functions

$$p_k(\zeta) = (1 - \zeta^2/m_k^2)^{-1/2}, \tag{15}$$

are analytic in the complex plane $\zeta = \xi + \mathrm{i}\eta$ cut along $|\xi| < m_k$, $\eta = 0$ and in eqn (12) are evaluated in the same quadrant as ζ_k. The arbitrary functions Ω_k and ω_k are defined in the upper and lower half of the ζ_k-plane, respectively. In this way, the solution to any particular problem [such as eqns (1)–(7)] is now reduced to constructing these functions in their regions of definition so that, upon substitution of eqn (14), the boundary conditions are satisfied by eqns (10) and (11).

A further general result can now be obtained in view of a stress-free condition on the half-plane surface *outside* the contact zone, i.e. a condition like eqn (4c). Specifically, the latter requirement is automatically satisfied if in the upper half of the complex ζ-plane

$$R\Omega_1 = -Kp_2F - \mathrm{i}2G\,, \quad R\Omega_2 = Kp_1G - \mathrm{i}2F\,, \quad R \equiv 4 - K^2p_1p_2, \tag{16a–c}$$

while identical equations hold for the ω_k-functions in the lower half. The arbitrary functions $F(\zeta)$, $G(\zeta)$ are analytic everywhere except possibly along a line defined by the mapping eqn (14a) of the contact length in the (r, θ)-plane, i.e. $(-p < \xi < p, \eta = 0)$, where $p \equiv (\alpha/c_1)$ in the present problem. In eqn (16a), $R(\zeta)$ is the familiar *Rayleigh function* having a double root at $\zeta = 0$ and simple roots at $\zeta = \pm m_R \equiv \pm(c_R/c_1)$.

For insights into the appropriate behavior of the functions $F(\zeta)$ and $G(\zeta)$ in the ζ-plane, we note that a point at infinity corresponds to wavefronts in the physical plane, the origin corresponds to the apex, and that certain points along the Re(ζ)-axis correspond to the edges of the contact length and Rayleigh wavefronts. At these points, one has to take into account physical requirements like wavefront continuity, integrability of possible stress singularities and asymptotics at the punch-surface separation points. Thus, based also on their definitions in eqn (16) and symmetry about the origin, we conclude that $F(\zeta)$, $G(\zeta)$ must be finite at $\zeta = \pm m_R$ and should behave no worse than

$$F, G = 0(\zeta)\,; \quad F, G = 0(\zeta^2)\,; \quad F, G = 0(|\zeta \pm p|^{-n-\varepsilon}), \tag{17a}$$

as $\zeta \to \infty$, $\zeta \to 0$, and $|\zeta \pm p| \to 0$, respectively, where $0 < \varepsilon < 1$.

Finally, substitution of eqn (16) into eqns (10)–(15), shows that for $0 < r < c_1 t$, $\theta = 0$ and $\theta = -\pi$, the complex displacement and stress are given in terms of F, G by

$$u = \frac{1}{2c_1 n[(n-1)!]} \int_{r/c_1 t}^{1} \frac{(vt - r/c_1)^n}{v^{n+2}} \left[D\left(\frac{LF}{R} - \mathrm{i}p_1 \frac{v^2 G}{m^2 R}\right) + \mathrm{i}D\left(\frac{LG}{R} + \mathrm{i}p_2 \frac{v^2 F}{m^2 R}\right)\right] \mathrm{d}v, \tag{18}$$

$$s_\theta = \frac{\mu}{2c_1^2(n-1)!} \int_{r/c_1 t}^{1} \frac{(vt - r/c_1)^{n-1}}{v^{n+2}} [D(F) + \mathrm{i}D(G)]\, \mathrm{d}v, \tag{19}$$

where

$$D() = \mathrm{Re}\,[()^+ - ()^-]\,, \quad f^\pm(\zeta) \equiv f(\xi \pm \mathrm{i}0)\,, \quad f^\pm = f(v \pm \mathrm{i}0) \quad \text{for } \theta = 0,$$
$$f^\pm = f(-v \pm \mathrm{i}0) \quad \text{for } \theta = -\pi, \tag{20}$$

$$L = 2 - Kp_1 p_2. \tag{21}$$

The solution for any particular problem requires the determination of F and G in view of the insights noted above. This can be done by imposing the boundary conditions and solving mixed BVPs in the ζ-plane. General forms for the surface displacement and stress, in terms of certain constants, can thus be obtained. Finally, a displacement matching procedure based on the imposed displacements in the contact region provides the values of the aforementional constants, thereby completing the solution.

BASIC SOLUTION

The $F(\zeta)$, $G(\zeta)$ functions for the problem (1)–(7) (wedge indentation, $n = 1$) will be obtained in this section.

In view of eqns (9c) and (19), eqn (4b) is automatically satisfied if

$$\frac{\mathrm{Re}\,(F^+ - F^-)}{\mathrm{Re}\,(G^+ - G^-)} = \frac{1}{\mathrm{sgn}\,(\mathrm{Re}\,\zeta)\cdot\gamma} \quad \text{for } -p < \mathrm{Re}\,\zeta < p, \quad \mathrm{Im}\,\zeta \to 0, \tag{22}$$

which defines a *Plemelj problem*, with the following solution (Gakhov, 1966; Roos, 1969)

$$G(\zeta) = \mathrm{sgn}\,(\mathrm{Re}\,\zeta)\cdot\gamma\cdot F(\zeta). \tag{23}$$

Now, in view of the general form of the u-displacement in eqn (18) and with the help of eqn (23), eqn (7) is satisfied if and only if

$$F^+(\zeta) = C(\zeta)\cdot F^-(\zeta) \quad \text{for } -p < \mathrm{Re}\,\zeta < p, \quad \mathrm{Im}\,\zeta \to 0, \tag{24}$$

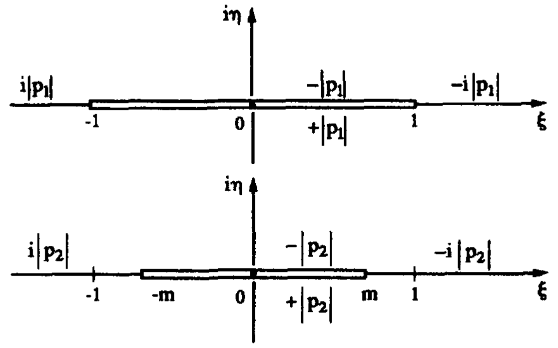

Fig. 2. Branch cuts for the functions $p_k(\zeta)$ in the complex ζ-plane.

where

$$C(\zeta) = \frac{[A(\zeta)+\mathrm{i}B(\zeta)]^-}{[A(\zeta)+\mathrm{i}B(\zeta)]^+}, \quad C(\xi) = \frac{A^-(\xi)+\mathrm{i}B^-(\xi)}{A^-(\xi)-\mathrm{i}B^-(\xi)}, \tag{25a, b}$$

$$A(\zeta) = \frac{\operatorname{sgn}(\operatorname{Re}\zeta)\cdot(\beta+\gamma)\cdot L(\zeta)}{R(\zeta)}, \quad A^-(\zeta) = \operatorname{sgn}(\xi)\cdot(\beta+\gamma)\cdot\frac{2-K|p_1|\,|p_2|}{4-K^2|p_1|\,|p_2|}, \tag{26a, b}$$

$$B(\zeta) = \frac{\zeta^2[p_2(\zeta)-\beta\cdot\gamma\cdot p_1(\zeta)]}{m^2 R(\zeta)}, \quad B^-(\xi) = \frac{\xi^2[|p_2|-\beta\gamma\cdot|p_1|]}{m^2(4-K^2|p_1|\,|p_2|)}. \tag{27a, b}$$

The results in eqns (25b), (26b) and (27b) were obtained by considering limiting values of the functions involved, as $\mathrm{Im}\zeta \to \pm 0$ along the branch cuts defined by $p_k(\zeta)$ and shown in Fig. 2.

Equation (24) defines a homogeneous *Riemann–Hilbert* problem (Gakhov, 1966; Roos, 1969), the solution of which may provide the as yet unknown function $F(\zeta)$. Indeed, it is obtained as

$$\begin{aligned} F(\zeta) &= P_m(\zeta)\cdot\exp\left[(2\pi\mathrm{i})^{-1}\int_{-p}^{p}\frac{\ln[C(v)]}{v-\zeta}\,\mathrm{d}v\right] \\ &= P_m(\zeta)\cdot\exp\left[\pi^{-1}\int_{-p}^{p}\frac{\tan^{-1}[\kappa^*(v)]}{v-\zeta}\,\mathrm{d}v\right], \end{aligned} \tag{28}$$

where $P_m(\zeta)$ is a polynomial (entire function) determined by the required asymptotic behavior in eqn (17), and the function $\kappa^*(v)$ is found from eqn (25b) as

$$\kappa^*(v) = \frac{B^-(v)}{A^-(v)} \equiv \operatorname{sgn}(v)\cdot\kappa(v), \quad \kappa(v) = \frac{v^2(|p_2|-\beta\gamma\cdot|p_1|)}{m^2(\beta+\gamma)(2-K|p_1|\,|p_2|)}, \tag{29a, b}$$

with $\kappa(0) = \dfrac{\beta\gamma-1}{m^2(\beta+\gamma)}$.

An immediate observation, however, in eqns (24)–(29) is that *Hölder continuity* does not prevail since $\tan^{-1}[\kappa^*(v)]$ jumps from a certain value to the opposite one when v crosses zero. Accordingly, the underlying theory for the Riemann–Hilbert problem cannot be utilized for the interval $(-p < \xi < p, \eta = 0)$, but one has to consider a two-part line of discontinuity [in the sense of eqn (24)], i.e. $(-p < \xi < 0, \eta = 0)$, $(0 < \xi < p, \eta = 0)$, where the $\kappa(v)$ function is now Hölder continuous. Thus, eqn (28) is written as

$$F(\zeta) = P_m(\zeta)\cdot\exp\left[-\pi^{-1}\int_{-p}^{0}\frac{\tan^{-1}[\kappa(v)]}{v-\zeta}\,dv+\pi^{-1}\int_{0}^{p}\frac{\tan^{-1}[\kappa(v)]}{v-\zeta}\,dv\right] \equiv P_m(\zeta)\cdot H(\zeta). \tag{30}$$

Following now the usual procedure for this class of problems, one must determine the behavior of the exponential term near the end points $\xi = -p, 0, p$; $\eta = 0$ in order to check whether the physical expectations for the mathematical solution are fulfilled. Otherwise, this so-called fundamental solution should be modified. Here, this procedure is carried out by invoking a well-known property of Cauchy integrals [see e.g. Gakhov (1966), p. 53]. To this end, by introducing the new function $\Omega(\xi)$ as

$$\Omega(\xi) \equiv -\frac{1}{\pi}\tan^{-1}\left[\frac{1}{\kappa(\xi)}\right] = \frac{1}{\pi}\tan^{-1}\left[\frac{-m^2(\beta+\gamma)(2-K|p_1|\,|p_2|)}{\xi^2(|p_2|-\beta\gamma\cdot|p_1|)}\right], \tag{31}$$

where $0 \leqslant \Omega(\xi) \leqslant 1/2$ for $\gamma > |\beta| \equiv \cot\Theta$, $-1/2 \leqslant \Omega(\zeta) \leqslant 0$ for $\gamma < |\beta| \equiv \cot\Theta$, and $\Omega = 0$ for $\gamma = |\beta| \equiv \cot\Theta$ or for both a *frictionless* and *uncoupled* problem, the exponential term in eqn (30) becomes

$$H(\xi) = |\xi|^{1-2\Omega(\xi)}|p^2-\xi^2|^{-1/2+\Omega(\xi)}\exp\left[b(\xi)\pm i\pi[\tfrac{1}{2}-\Omega(\xi)]\right] \quad \text{for } 0<\xi<p, \quad \eta\to\pm 0, \tag{32}$$

where $(-\pi)$ replaces (π) above, for $-p < \xi < 0$, and

$$b(\xi) = \int_{-p}^{p}\operatorname{sgn}(v)\frac{\Omega(v)-\Omega(\xi)}{v-\xi}\,dv = 2\int_0^p\frac{\Omega(v)-\Omega(\xi)}{v^2-\xi^2}v\cdot dv. \tag{33}$$

In the same manner [see e.g. Gakhov (1966)], we find that the fundamental function is

$$H(\zeta) = \zeta^{1-2\Omega(\xi_0)}(\zeta^2-p^2)^{-1/2+\Omega(\xi_0)}\,e^{\Lambda(\zeta,\xi_0)}, \tag{34}$$

where

$$\Lambda(\zeta,\xi_0) = \int_{-p}^{p}\operatorname{sgn}(v)\frac{\Omega(v)-\Omega(\xi_0)}{v-\zeta}\,dv, \tag{35}$$

and ξ_0 is an arbitrary point on the interval $[-p,p]$ which can be chosen as $\xi_0 = 0$ for convenience in calculations presented below. Consequently, eqn (34) becomes

$$H(\zeta) = \zeta^{1-2\Omega_0}(\zeta^2-p^2)^{-1/2+\Omega_0}\,e^{\Lambda_0(\zeta)}, \tag{36}$$

with

$$\Omega_0 \equiv \Omega(\xi_0=0) = \frac{1}{\pi}\tan^{-1}\left[\frac{m^2(\beta+\gamma)}{1-\beta\gamma}\right], \tag{37}$$

$$\Lambda_0(\zeta) \equiv \Lambda(\zeta,\xi_0=0) = \int_{-p}^{p}\operatorname{sgn}(v)\frac{\Omega(v)-\Omega_0}{v-\zeta}\,dv. \tag{38}$$

The first term in the r.h.s. of (34) reflects the absence of Hölder continuity in eqn (24) coming from both the changeover of boundary condition (4b) in the physical problem and the displacement-coupling formulation expressed by eqn (7).

Now, by taking also into account the asymptotic conditions [eqn (17)], it is concluded that the full solution to eqn (24) is

$$F(\zeta) = -\mathrm{i}Q\cdot\zeta\cdot H(\zeta), \tag{39}$$

where Q is a real constant, which will be determined later on through a *displacement matching* procedure. Several important results, however, can be obtained without specific knowledge of that as shown in the next section.

CONTACT STRESS AND RESULTANT NORMAL FORCE

In this section, general expressions are derived for the contact-stress distribution and the resultant normal force exerted on the contact zone. These results can be obtained prior to the evaluation of the constant Q and yet exhibit important aspects of the physical problem.

Starting with the normal contact stress ($r < \alpha t$), eqns (9c), (19) and (39) when combined give

$$\sigma_\theta(r,\theta=0,t) = -\frac{Q\mu}{c_1^2}\cdot\int_{r/c_1 t}^{p} e^{h(v)}\cos[\pi\Omega(v)]\cdot\left(\frac{v^2}{p^2-v^2}\right)^{1/2-\Omega(v)}\frac{\mathrm{d}v}{v^2}. \tag{40}$$

Symmetry, of course, implies that $\sigma_\theta(r,\theta=-\pi,t)$ is also given by eqn (40). Thus, the contact stress can be obtained from the latter expression by simple *numerical* integration. Also, an immediate observation is that at the contact-zone end ($r = \alpha t, \theta = 0$), i.e. as $r/c_1 t \to p$, the stress vanishes and, therefore, the solution exhibits the expected smooth transition from contact to separation.

It is of interest, next, to extract from eqn (40) the contact-stress behavior at the wedge-apex station (origin). As $r \to 0$, depending upon the interval where $\Omega(v)$ takes values, i.e. depending upon the relative magnitude of the friction coefficient γ and the coupling coefficient $|\beta| \equiv \cot\Theta$, the integral may become *regular*, *singular* or *hypersingular*, and therefore the following three cases are distinguished:

(a) For $\gamma < |\beta| \equiv \cot\Theta$ and $0 < v < p$, eqn (31) implies that $-1/2 < \Omega(v) < 0$, and the integral in eqn (40) is regular when $r/c_1 t = 0$. Thus, the contact stress at the wedge apex station will be *finite* in this case where the displacement-coupling effect is stronger than the friction-reversal effect. The same conclusion also applies in the absence of friction corroborating in this way earlier results obtained by Brock (1979).

(b) For $\gamma = |\beta| \equiv \cot\Theta$ or for *both* a *frictionless* and *uncoupled* formulation of the problem, eqn (31) implies that $\Omega(v) = 0$, and the integral in eqn (40) is singular at $r/c_1 t = 0$. An asymptotic evaluation gives

$$\sigma_\theta(r\to 0,\theta=0,t) \cong -\frac{Q}{c_1}\frac{\mu}{\alpha}\cdot\ln\left[\frac{\alpha t}{r}+\left(\frac{\alpha^2 t^2}{r^2}-1\right)^{1/2}\right]+\ldots, \tag{41}$$

thus demonstrating a wedge-apex logarithmic singularity in contact stress. Of course, in the *classical* formulation of the wedge indentation problem with *no* friction and *no* coupling effects, this is a well-known result [see e.g. Eringen and Suhubi (1975); Bedding and Willis (1976)], but it is noteworthy that within the present formulation this also appears for the limit case where $\gamma = \cot\Theta$.

(c) For $\gamma > |\beta| \equiv \cot\Theta$ and $0 < v < p$, eqn (31) implies that $0 < \Omega(v) < 1/2$, and the integral in eqn (40) is hypersingular at $r/c_1 t = 0$. However, an asymptotic evaluation can still be performed according to the following general result (Hadamard, 1923)

$$\lim_{\tau \to x}\left[\int_x^a \frac{f(\tau)}{(\tau - x)^\phi}\,\mathrm{d}\tau\right] = \lim_{\varepsilon \to 0}\left[\frac{f(x)}{(\phi - 1)\cdot \varepsilon^{\phi - 1}}\right], \quad \text{with } 1 < \phi. \tag{42}$$

Indeed, in view of eqn (42), eqn (40) gives

$$\sigma_\theta(r \to 0, \theta = 0, t) \cong -\frac{Q\mu \cdot \exp(S_0)\cdot \cos(\pi\Omega_0)}{2\Omega_0 c_1 \alpha}\left(\frac{\alpha t}{r}\right)^{2\Omega_0} + \ldots, \tag{43}$$

where Ω_0 is already given in eqn (37), and

$$S_0 \equiv b(\xi = 0) = 2\int_0^p \frac{\Omega(w) - \Omega_0}{w}\,\mathrm{d}w. \tag{44}$$

It should be noted that in this case, where the friction-reversal effect is stronger than the displacement-coupling effect, the contact stress exhibits a *power-type* singularity of order $2\Omega_0 < 1$. This result is in agreement with previous ones obtained for *uncoupled* elastostatic and elastodynamic indentations in the presence of friction reversal (Roberts, 1970; Brock, 1993; Brock and Georgiadis, 1994; Brock *et al.* 1993, 1994). The physical explanation of this clue stems from the fact that a distributed-loading discontinuity at 0 occurs [as eqn (4b) implies] and, thus, the origin 0 is a point of *flux* singularity (Erdogan, 1978). At the same point also, a *geometrical* discontinuity occurs and this is responsible for the logarithmic singularity encountered within the frictionless and uncoupled problem formulation.

Finally, it is of interest to observe that the singular contact-stress behavior in eqn (43) suggests the possibility of surface cracking beneath the indentor, and thus the present results may also be useful in fracture mechanics.

Now attention is directed to the determination of the *normal force P* on the contact zone ($r < \alpha t$; $\theta = 0, -\pi$). Notice that this external force required to drive the indentor into the deformable medium is unknown at the outset and has to be determined from the solution. This is in some contrast to static contact problems, where the external force is prescribed in the analysis. Here, however, it is the indentor velocity which is prescribed reflecting the difficulties encountered and the assumptions made in solving analytically transient contact problems. The P force follows from the relation

$$P = -2\int_0^{\alpha t} \sigma_\theta(r, \theta = 0, t)\,\mathrm{d}r = -2c_1 t\cdot \int_0^p \sigma_\theta(w, \theta = 0)\,\mathrm{d}w, \tag{45}$$

where σ_θ is given by eqn (19) for a general self-similar indentation. Substitution of eqn (19) in eqn (45) along with an interchange of integrations leads to the following general result:

$$P = -\frac{\mu t^n}{c_1 n!}\cdot \int_0^p \frac{\mathrm{Re}\,(F^+ - F^-)}{v^2}\,\mathrm{d}v. \tag{46}$$

In our case, where $F(\zeta)$ is given by eqn (39), the above equation yields

$$P = \frac{\pi\mu Q}{c_1}t. \tag{47}$$

From its definition in eqn (45), P should be a positive quantity and therefore Q must also be positive. This, in fact, is verified by the numerical results. Accordingly, the normal stress given by eqn (40) is appropriately *compressive* in all points of the contact zone.

FURTHER ELABORATION OF THE SOLUTION

Here, we obtain the as yet unknown constant Q appearing in the solution and also an interrelation of the problem parameters α, V and Θ. Both results are found by matching the imposed contact displacement [given by eqn (7)] with the general form [eqn (18), incorporating the basic solution eqn (39)] predicted by the mathematical analysis. The latter analysis gives the following expressions for the contact-zone displacements ($r < \alpha t$):

$$u_\theta(r, \theta = 0, t) = \frac{1}{2c_1}\cdot\int_{r/c_1 t}^{p}\frac{(vt - r/c_1)}{v^3}\cdot \mathrm{Re}\,[(X+\mathrm{i}Y)^+\cdot F^+ - (X+\mathrm{i}Y)^-\cdot F^-]\,\mathrm{d}v$$
$$+\frac{1}{2c_1}\cdot\int_{p}^{1}\frac{(vt - r/c_1)}{v^3}\cdot \mathrm{Re}\,[((X+\mathrm{i}Y)^+ - (X+\mathrm{i}Y)^-)\cdot\hat{F}]\,\mathrm{d}v, \tag{48}$$

$$u_r(r, \theta = 0, t) = \frac{1}{2c_1}\cdot\int_{r/c_1 t}^{p}\frac{(vt - r/c_1)}{v^3}\cdot \mathrm{Re}\,[(S-\mathrm{i}T)^+\cdot F^+ - (S-\mathrm{i}T)^-\cdot F^-]\,\mathrm{d}v$$
$$+\frac{1}{2c_1}\cdot\int_{p}^{1}\frac{(vt - r/c_1)}{v^3}\cdot \mathrm{Re}\,[((S-\mathrm{i}T)^+ - (S-\mathrm{i}T)^-)\cdot\hat{F}]\,\mathrm{d}v, \tag{49}$$

where

$$X \equiv \frac{\gamma L}{R}, \quad Y \equiv \frac{p_2 v^2}{m^2 R}, \tag{50a, b}$$

$$S \equiv \frac{L}{R}, \quad T \equiv \frac{\gamma p_1 v^2}{m^2 R}, \tag{51a, b}$$

$$\hat{F}(v) = -\mathrm{i}Q\cdot v^{2-2\Omega_0}(v^2 - p^2)^{-1/2+\Omega_0}\,\mathrm{e}^{\Lambda_0(v)}, \tag{52}$$

and $F^+(v)$, $F^-(v)$ result by combing eqns (32) and (39).

Equations (48) and (49) can take explicit integral forms by considering the behavior of the $p_k(\zeta)$ functions along pertinent intervals of the real axis in the *cut* complex-plane (see Fig. 2). Also, surface particle velocities can readily result by applying the Leibnitz rule to these expressions.

Now, by using eqns (48) and (49) in eqn (7) and by taking also into account eqn (24), one obtains the following algebraic system, which provides Q and the interrelation of the problem parameters α, V and Θ

$$Q\cdot(I_1 - I_2) - \beta\cdot Q\cdot(I_5 + I_6) = c_1 V, \tag{53a}$$

$$Q\cdot(I_3 - I_4) - \beta\cdot Q\cdot(I_7 + I_8) = c_1^2\cdot\cot\Theta. \tag{53b}$$

where

$$I_1 = \fint_{p}^{m} v^{-2\Omega_0}(v^2 - p^2)^{-1/2+\Omega_0}\,\mathrm{e}^{\Lambda_0(v)}\,Y_a^-(v)\,\mathrm{d}v, \tag{54a}$$

$$I_2 = \int_{m}^{1} v^{-2\Omega_0}(v^2 - p^2)^{-1/2+\Omega_0}\,\mathrm{e}^{\Lambda_0(v)}\,Y_b^-(v)\,\mathrm{d}v, \tag{54b}$$

$$I_3 = ⨍_p^m v^{-1-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, Y_a^-(v)\, dv, \tag{54c}$$

$$I_4 = \int_m^1 v^{-1-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, Y_b^-(v)\, dv, \tag{54d}$$

$$I_5 = ⨍_p^m v^{-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, T_a^-(v)\, dv, \tag{55a}$$

$$I_6 = \int_m^1 v^{-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, T_b^-(v)\, dv, \tag{55b}$$

$$I_7 = ⨍_p^m v^{-1-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, T_a^-(v)\, dv, \tag{55c}$$

$$I_8 = \int_m^1 v^{-1-2\Omega_0}(v^2-p^2)^{-1/2+\Omega_0}\, e^{\Lambda_0(v)}\, T_b^-(v)\, dv, \tag{55d}$$

and

$$Y_a^- = \frac{|p_2|\cdot v^2}{m^2(4-K^2|p_1|\cdot|p_2|)}, \tag{56a}$$

$$Y_b^- = \frac{m^{-2}v^2K\cdot|p_1|\cdot(K\cdot p_2^2-2\gamma\cdot|p_2|)}{16+K^4p_1^2p_2^2}, \tag{56b}$$

$$T_a^- = \frac{\gamma\cdot|p_1|\cdot v^2}{m^2(4-K^2|p_1|\cdot|p_2|)}, \tag{57a}$$

$$T_b^- = \frac{2m^{-2}v^2|p_1|\cdot(2\gamma-K\cdot|p_2|)}{16+K^4p_1^2p_2^2}. \tag{57b}$$

In eqns (54) and (55), $⨍$ signifies *Cauchy principal-value* integrals, since the *Rayleigh* pole at $v = m_R$ occurs in the respective integrands.

Finally, it is noted that in the *limit* case where displacement coupling is neglected, the coupling coefficient $\beta(\equiv -\cot\Theta)$ should be set equal to zero, and then the system (53) degenerates to the form given by Brock and Georgiadis (1993).

NUMERICAL RESULTS AND CONCLUDING REMARKS

Some numerical results are here presented and discussed. These results were obtained for characteristic wave velocities $c_1 = 1870.83$ *m/sec*, $c_2 = 1000.00$ *m/sec* and $c_R = 927.41$ *m/sec* ($m \equiv c_2/c_1 = 0.534$ and $m_R \equiv c_R/c_1 = 0.495$), wedge inclination angles $\Theta = 86°$ and $\Theta = 89°$, and several values of the friction coefficient γ.

The integrals in eqns (54) and (55) were evaluated numerically by the Gauss rule. Also, the Cauchy principal-value integration was performed almost in accordance with its definition, i.e. a very small integration interval symmetrically situated about $v = m_R$ is *excluded* with the idea the unbounded positive and negative areas to the left and right of the singularity cancel each other. In the latter procedure, the extent of the small excluded

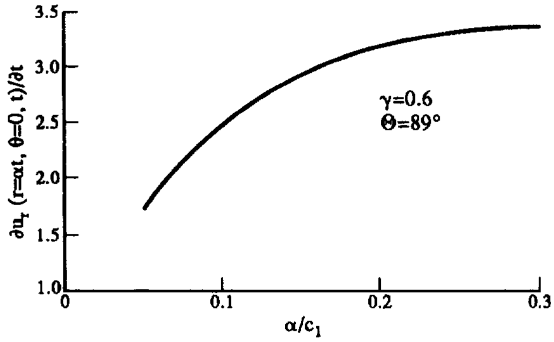

Fig. 3. Values of the tangential particle velocity at the contact-zone edge.

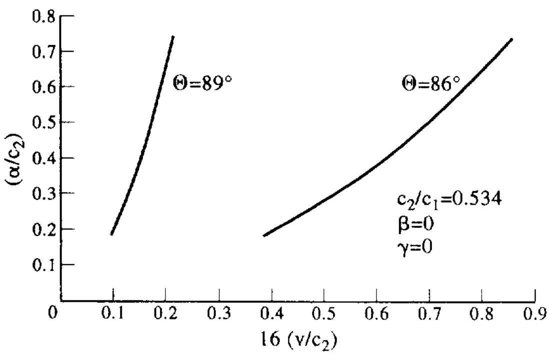

Fig. 4. Relation between the contact-zone expansion speed α and the indentor speed V for the classical (uncoupled and frictionless) wedge-indentation problem.

segment is determined by decreasing its length until the *first* divergence in the results occurs. This technique was tested and proved very accurate against special Cauchy principal-value integration algorithms, for the case of otherwise smooth integrands, in another recent work of ours (Brock *et al.* 1994). Indeed, the integrands in eqns (54a),(54c), (55a) and (55c) are smooth and monotone functions when the interval near the pole singularity is excluded.

First, the sign of $\partial u_r(r < \alpha t, \theta = 0, t)/\partial t$ was checked over several values of the contact-zone expansion speed α. It was always found that $\dot{u}_r > 0$ along the contact zone (except, of course, at $r = 0$ where $\dot{u}_r = 0$ due to symmetry). Thus, the constraint in eqn (8) is satisfied. Figure 3, for instance, shows values (positive) of the tangential particle velocity at the edge of the contact zone ($r = \alpha t$) for $\Theta = 89°$, $\gamma = 0.6$ and variable α. The latter quantity was obtained from eqn (49) as $\partial u_r(r = \alpha t, \theta = 0, t)/\partial t = (Q/c_1)[(I_5+I_6)-(I_7+I_8)(\alpha/c_1)]$. One can clearly observe in Fig. 3 that there is a tendency of the elastic half-plane material to slide out (in the tangential direction) of the contact zone (away from the wedge apex), and this result is in accordance with other recent results for the uncoupled frictional problem (Brock, 1993; Brock and Georgiadis, 1994).

Next, another check on the present solution is provided by numerically obtaining the relation between the normalized contact-zone expansion velocity (α/c_2) and the normalized indentor velocity (V/c_2), for the uncoupled ($\beta = 0$) and frictionless ($\gamma = 0$) case. Indeed, our results in Fig. 4 agree with the results obtained by a different solution technique for the latter classical formulation of the problem (Eringen and Suhubi, 1975). However, it is noticeable that the $\alpha - V$ relation remains almost the same for the present non-classical problem as well. For instance, Fig. 5 shows the $\alpha - V$ relation for two different half-wedge angles ($\Theta = 86°$ and $\Theta = 89°$) in a problem formulation with coupling ($\beta = -\cot\Theta$) and frictional ($\gamma = 0.4$) effects.

Finally, Figs 6 and 7 illustrate the distribution of the normalized contact-zone normal stress $\sigma_\theta(r \leqslant \alpha t, \theta = 0, t)$ along the normalized contact region ($r/c_1 t$) for $p \equiv \alpha/c_1 = 0.3$. Two different values of γ were considered for $\Theta = 86°$ and $89°$. In the first case of *weak*

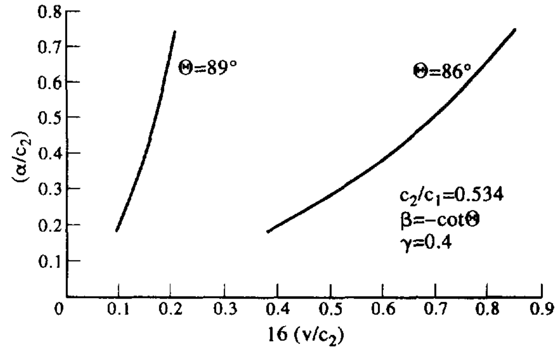

Fig. 5. Relation between the contact-zone expansion speed α and the indentor speed V for the non-classical (coupled and frictional) wedge-indentation problem.

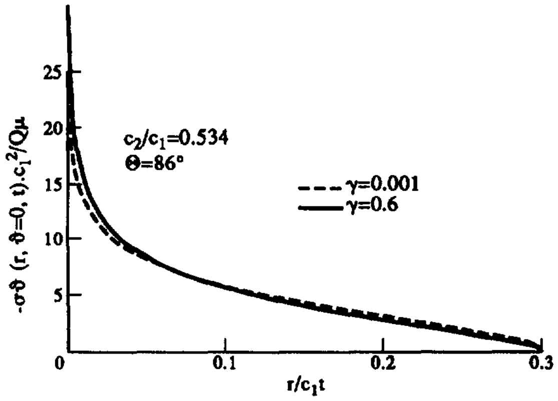

Fig. 6. Contact stress distributions for $\Theta = 86°$. The solution behavior at the wedge-apex station depends upon the relative magnitude of Θ and γ.

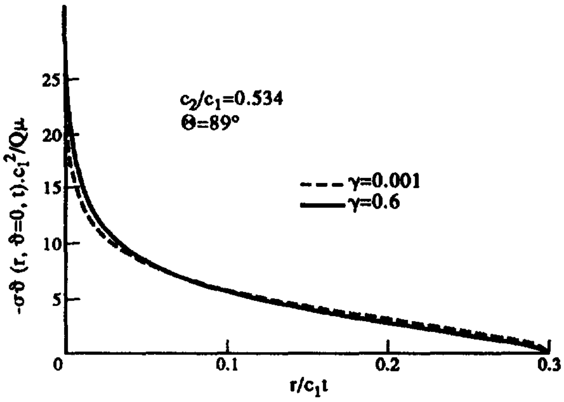

Fig. 7. Contact stress distributions for $\Theta = 89°$. The solution behavior at the wedge-apex station depends upon the relative magnitude of Θ and γ.

friction ($\gamma = 0.001$), the coupling effect dominates the frictional effect ($\gamma < |\beta| \equiv \cot \Theta$) and $\sigma_\theta(r \to 0, \theta = 0, t)$ is *finite* at the wedge-apex station. In the second case of *strong* friction ($\gamma = 0.6$), the frictional effect dominates the coupling effect ($\gamma > |\beta| \equiv \cot \Theta$) and $\sigma_\theta(r \to 0, \theta = 0, t)$ becomes *singular* at the apex. Also, Figs 6 and 7 depict that the contact stress does not depend appreciably upon the relative magnitude of β and γ in regions not too close to the apex station. However, the coupled formulation yields a more natural solution behavior with finite stress at every point of the contact zone.

In conclusion, the present exact study models certain realistic contact mechanics situations where a rigid wedge-shaped body rapidly indents a deformable half-plane. The solution is valid for an elastic half-plane and, thus, it remains valid only during the early time interval of the dynamic contact of the two bodies. Frictional and tangential-displacement effects are included, and the analysis shows that these have a significant influence on the contact-stress distribution. In particular, when a certain relation between the wedge inclination angle and the friction coefficient exists ($\cot \Theta > \gamma$), no *pathological* stress singularities occur within the contact zone and, therefore, the solution exhibits a more natural behavior as compared with the one given by the classical analysis of the problem.

One final comment is also in order. It is felt that, in this class of elastodynamic indentation problems, analytical solutions (like the present one) despite their obvious limitations are more advantageous than numerical solutions (obtained by finite differences, finite elements, or boundary elements) in some respects. For instance, (a) spurious waves may be generated by the discontinuities due to the discretization, and (b) the true solution may involve features on a scale smaller than the discretization used (e.g. the stress behavior at the apex vicinity in the present problem will require a very fine mesh for its resolution). Certain cases enforcing the foregoing statements have already been identified in a recent work by Georgiadis and Barber (1993), studying general existence/uniqueness issues of such elastodynamic indentation problems.

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